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Radiation-balanced simulations for binary inspiral

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Abstract

The late stage of the inspiral of two black holes may have important non-Newtonian effects that are unrelated to radiation reaction. To understand these effects we approximate a slowly inspiralling binary by a stationary solution to Einstein's equations in which the holes orbit eternally. Radiation reaction is nullified by specifying a boundary condition at infinity containing equal amounts of ingoing and outgoing radiation. The computational problem is then converted from an evolution problem with initial data to a boundary value problem. In addition to providing an approximate inspiral waveform via extraction of the outgoing modes, our approximation can give alternative initial data for numerical relativity evolution. We report results on simplified models and on progress in building 3D numerical solutions.

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1. Introduction

A pair of compact objects (black holes or neutron stars) in binary orbit will, according to Newtonian gravity, remain in the same orbits forever. In general relativity, however, the system will emit gravitational radiation, causing the bodies to spiral in towards one another. The gravitational radiation given off by this system is a prime candidate for detection by upcoming gravitational wave telescopes, such as LIGO and VIRGO [1].

Full three-plus-one numerical evolution of the Einstein equations is a powerful tool for determining the history of such a system and the form of the gravitational radiation emitted, but is limited in its applicability by instabilities which prevent current codes from evolving for more than an orbit or so. Thus it is not only efficient but also necessary to limit full-numerical simulations to the portion of the evolution for which there is no applicable approximation scheme [2]. At least some of the inspiral can be handled by post-Newtonian methods [3],

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while the final post-merger ringdown can be treated with the black-hole perturbation theory [4]. The purpose of the quasi-stationary approximation is to provide improved early waveforms and later-time supercomputer initial conditions by modelling a late inspiral phase, in which some nonperturbative gravitational effects are relevant, but the radiation-induced inspiral is still occurring slowly.

2. Quasi-stationary approximation

The idea, initially proposed by Blackburn and Detweiler [5], is that if the inspiral is slow, the system is nearly periodic: after one orbit, the objects have returned almost to their original locations, and radiation which has moved out has been replaced with new radiation of approximately the same shape. If the objects' orbits are circular rather than elliptical, the spacetime is nearly stationary. If the approximate orbital frequency is Ω, moving forward in time by δt and rotating the resulting spatial slice by $-\Omega\ \delta t$ will not change the picture very much.

This approximation can be used to simplify the numerical problem by solving for an exactly stationary spacetime which serves, over some period of time, as a reasonable approximation for the slowly evolving spacetime. In the process, the three-plus-one dimensional evolution problem is reduced to a three-dimensional, nondynamical one, not only reducing the size of the computational grid, but also hopefully avoiding evolutionary instabilities.

Since gravitational radiation emitted from the orbiting system carries away energy, some modification must be made to the physical problem in order to allow this equilibrium solution. One approach [6] is to satisfy only a subset of the Einstein equations, typically only the constraint equations. This is often combined with a requirement that the spatial geometry of the stationary spacetime be conformally flat, a simplification which can be thought of as suppressing the radiative degrees of freedom in Einstein's theory, the waves that would carry off energy from orbiting objects. Our approach, on the other hand, is to keep the gravitational radiation, but nullify the radiation reaction by balancing the outgoing radiation by an equal amount of incoming radiation. In doing so, we will solve the full Einstein equations in the presence of the actual sources, and simply replace the physical boundary condition of outgoing radiation at large distances with one of balanced radiation.

3. Radiation-balanced boundary conditions

In the general context of a theory in which a pair of orbiting sources for a wave equation give off radiation, it is instructive to consider three types of solutions. In a *type I* solution, the radiation is outgoing, carrying away energy and causing the orbits to decay. In a *type II* solution, the radiation is again outgoing, but some additional force, uncoupled to the radiating field, keeps the sources in their circular orbits, resulting in a stationary spacetime. A *type III* solution is also stationary, with the sources remaining in circular orbits of constant frequency, but this time there is no external force, and the equilibrium is maintained by a balance of incoming and outgoing radiation. Type I is the physical situation we ultimately wish to model. Type II should be a reasonable approximation to type I if the type I solution in question is inspiralling only slowly. It should not be appropriate to general relativity, in which all matter and energy couple to the gravitational field, but is useful for comparison in theories which allow for external forces. Type III is the stationary solution we wish to find numerically, and, where appropriate, to relate to a type II or I solution.

3.1. Scalar field theory results

To analyse in detail the implications of boundary conditions on radiating, stationary solutions (types II and III), prior work on this project [7] has considered the theory of a nonlinear scalar field $\psi(t, r, \theta, \phi)$ in Minkowski spacetime, the simplest theory with nonlinearity, orbits and radiation. If the source is required to rotate at a constant angular velocity Ω, so that the charge density $\rho(t, r, \theta, \phi)$ is a function only of r, θ, and $\varphi = \phi - \Omega t$, there exist stationary solutions for which the field exhibits the same symmetry; in that case, the wave equation can be written as

$$\left(\nabla^2 - \Omega^2 c^{-2} \partial_\varphi^2\right) \psi = -\rho + \lambda \mathcal{F}(\psi) \tag{1}$$

where ∇^2 is the spatial Laplacian and $\mathcal{F}(\psi)$ is a nonlinear coupling. Convenient choices of charge distribution include equal and opposite point charges at antipodal points in the equatorial plane, or translationally invariant line charges perpendicular to the equatorial plane. (The latter choice renders the problem equivalent to that of point charges in 2 + 1 dimensions.)

In the context of a numerical solution on a finite grid, the type II (purely outgoing radiation) solution $\psi_{\rm out}^R$ is defined by applying a Sommerfeld $\left(\psi_{,r} + c^{-1}\psi_{,t} = \psi_{,r} - \Omega c^{-1}\psi_{,\varphi} = 0\right)$ boundary condition at the radius R of the outer boundary of the grid. If the nonlinear term becomes small at large distances, the form of any solution near a large-R boundary will be a solution to the vacuum, linear version of (1); each angular Fourier mode will be a linear combination of two independent solutions, which can be identified as a purely ingoing and a purely outgoing solution. The limit $\psi_{\rm out} := \lim_{R\to\infty} \psi_{\rm out}^R$ is simply the solution consisting only of outgoing modes, with the coefficients of all the ingoing modes set to zero. (The entire discussion can be repeated for ingoing radiation, with the resulting solution being $\psi_{\rm in}(r, \theta, \varphi) = \psi_{\rm out}(r, \theta, -\varphi)$.)

The most familiar type III radiation-balanced (RB) solutions are standing-wave solutions in which the Dirichlet ($\psi = 0$) or Neumann ($\psi_{,r} = 0$) boundary condition is enforced at $r = R$. If the large-distance form of any of these solutions is resolved in angular Fourier modes, it is found that the amplitudes of the coefficients of the ingoing and outgoing components are equal; however, the relative phase of the two independent vacuum solutions depends on the choice of R, as does the radiation amplitude corresponding to a given source strength. Unlike outgoing- (or ingoing) wave solutions, these standing-wave solutions do not tend towards any limit as the boundary radius is taken to infinity.

Waves satisfying Neumann or Dirichlet boundary conditions are not the only solutions with equal magnitudes of incoming and outgoing radiation, and in fact are not the type that are appropriate for our problem. In the case of a linear theory ($\lambda = 0$), the relevant standing wave solution is simply a linear superposition of the ingoing and outgoing solutions (LSIO) $\psi_{\rm LSIO} = \frac{1}{2}(\psi_{\rm in} + \psi_{\rm out})$. For the nonlinear problem we want an analogue of this superposition. To this end we note that in the linearized theory, the LSIO solution is the RB solution with the minimum radiation in the wave zone. This means, in effect, that the LSIO is the solution with just the radiation needed to keep the sources in equilibrium and no 'extra' radiation. From the point of view of using a type III solution to approximate a type II one, we can take this 'minimum energy radiation balance' (MERB) solution, extract its outgoing component and identify it with the outgoing radiation.

In the nonlinear theory, the superposition of two solutions is no longer a solution. In [7], we reformulated (1) (in 2 + 1 dimensions, or equivalently with infinite line charges) as a Green's function problem and used the average of the advanced and retarded Green's functions to find a radiation-balanced solution. We found that even for highly nonlinear theories, this time-symmetric Green's function method can be used to find a good approximation for the outgoing solution. The success of the approximation means that superposition of the ingoing and outgoing solutions is approximately valid even when nonlinear effects are strong. The explanation for this is crucial to the physical basis for our approximation: in the innermost regions the fields are strong, but are very insensitive to the boundary conditions; the fields here differ only slightly when outgoing boundary conditions are changed to ingoing boundary conditions. In the wave zone region, the fields depend very sensitively on the boundary conditions, but here the fields are weak. Thus superposition approximately works because the solutions being superposed are almost the same where nonlinearities are strong, and are almost linear where the solutions are very different. The success of the numerical verification of this principle lends confidence that in the case of general relativity, if we can find a type III solution with little or no 'superfluous' radiation, we may be able to relate it to the physical type I solution in an analogous way.

As an aside, it is worth stressing that while equation (1) is elliptical inside and hyperbolic outside the 'speed of light cylinder' $r\sin\theta = 1/\Omega$, we experienced no difficulties in finding the numerical solution to the problem using closed-surface boundary conditions usually associated with a purely elliptical equation. In particular there were no discernable artefacts at the light cylinder, which required no special treatment in the numerical solution.

3.2. A more general prescription

Theories like GR, which have more involved nonlinearities than (1), cannot be reduced to the linear Green's function problem described in section 3.1. It is therefore necessary to develop a more general method for both imposing the condition of radiation balance and selecting an MERB solution.

Far from the sources, (1) reduces to

$$r^{-1}\partial_r^2\left(r\psi_\ell^m\right) + m^2\Omega^2\psi_\ell^m \to 0 \tag{2}$$

where we have resolved $\psi(r,\theta,\varphi) = \sum_{\ell m}\psi_\ell^m(r)Y_\ell^m(\theta,\varphi)$ in spherical harmonics. The general solution in the wave zone will then be

$$\psi_\ell^m \to r^{-1}\left(A_\ell^m \mathrm{e}^{\mathrm{i}m\Omega r} + B_\ell^m \mathrm{e}^{-\mathrm{i}m\Omega r}\right). \tag{3}$$

Since $Y_\ell^m(\theta,\varphi)$ includes a factor of $\mathrm{e}^{\mathrm{i}m\varphi}$ and hence $\mathrm{e}^{-\mathrm{i}m\Omega t}$, the first term in (3) represents an outgoing solution and the second an ingoing one. The condition of radiation balance is that, for each (ℓ, m) mode, A_ℓ^m and B_ℓ^m have the same amplitude. In other words, we can write $A_\ell^m/B_\ell^m = \exp\left(2\mathrm{i}\delta_l^m\right)$, where δ_l^m is an unspecified phase. (The possible choices of δ_l^m parametrize the radiation-balanced family of solutions.) Combining this with the form (3) of the solution in the wave zone, a particular radiation-balanced solution will obey

$$\frac{\partial_r\psi_\ell^m}{\psi_\ell^m} = \mathrm{i}m\Omega\frac{\exp\left[\mathrm{i}\left(m\Omega r + \delta_\ell^m\right)\right] - \exp\left[-\mathrm{i}\left(m\Omega r + \delta_\ell^m\right)\right]}{\exp\left[\mathrm{i}\left(m\Omega r + \delta_\ell^m\right)\right] + \exp\left[-\mathrm{i}\left(m\Omega r + \delta_\ell^m\right)\right]}. \tag{4}$$

The family of boundary conditions (4) allows us to construct the family of radiation-balanced solutions. To find the MERB solution, we just need to vary δ_ℓ^m to minimize the total 'energy' $2\sum_{\ell m}\left|A_\ell^m\right|^2$.

The actual numerical solution of (1) is a boundary value problem outside a mixed hyperbolic–elliptic region. The standing-wave boundary conditions for the nonlinear problem, furthermore, are very intricate since they require a three-step process of multipole decomposition, application of the boundary condition to each multipole and multipole summation. This multipole decomposition/recomposition is easier to implement as one step of an iterative relaxation process. Unfortunately, *standard* relaxation techniques will only work for a problem that is purely elliptic inside the boundary. This is especially unfortunate since relaxation methods are better suited to handling the numerically intensive work that will be necessary for our gravitational three-dimensional problem. For this reason we have developed a modified relaxation method that is not limited to elliptic problems. This method has been applied to the gravitational test problem of a single black hole. The numerical solution in fact has been found to converge to the correct solution, but quite slowly. A major focus of the project now is improved numerical algorithms for the boundary value problem.

4. Stationary rotating solutions in general relativity

In the theory of a scalar field in Minkowski space, the stationarity condition obeyed by type II and III solutions was that the field depended on the coordinates t and ϕ only in the combination $\varphi = \phi - \Omega t$, or equivalently, that $(\partial_t + \Omega\partial_\phi)\psi = 0$. In general relativity, this condition becomes a Killing symmetry $\mathcal{L}_K g_{ab} = 0$, where the Killing vector can be thought of as $K \sim \partial_t + \Omega\partial_\phi$.

An elegant way to impose the Killing symmetry is the Geroch decomposition [8], in which the four-geometry is described in terms of a set of scalar and vector fields on a three-dimensional manifold of Killing vector trajectories, plus the three-metric of this manifold. However, since our Killing vector is timelike near the axis of rotation and spacelike far from the axis (becoming null at the light cylinder), the manifold of Killing vector orbits has a surface of signature change, which makes the three-manifold of Killing vector orbits a problematic starting point for numerical simulation of this spacetime.

The approach currently being pursued is instead to perform the simulation in harmonic coordinates $\{x^i = t, x, y, z\}$, general analogues to Cartesian coordinates, which are annihilated by the spacetime d'Alembertian when treated as scalar fields. In analogy to rotating coordinates in flat space, we define $\xi = x\cos\Omega t + y\sin\Omega t$ and $\eta = -x\sin\Omega t + y\cos\Omega t$, and requires that $(\partial/\partial t)_{\xi,\eta,z}$ be a Killing vector. This means that in $\{t, \xi, \eta, z\}$ coordinates, the metric components only depend on ξ, η and z. The induced functional dependence of the metric components in the harmonic coordinate system is more complicated, but ultimately the unknown functions for which we numerically solve depend on only three coordinates. The Einstein equations, along with the harmonic gauge condition, can then be solved on the hypersurface $t = 0$, which completely determines the stationary spacetime.

5. Conclusions and outlook

This paper has described an ongoing research programme to find numerically a stationary spacetime which can approximate over some stretch of time a slowly inspiralling compact object binary. A stationary solution to the radiative problem is to be achieved by replacing the physical boundary condition of purely outgoing radiation with a condition corresponding to an equal mix of ingoing and outgoing radiation.

Numerical work so far has mostly focused on the properties of radiation-balanced solutions in scalar field theories (extracting outgoing radiation as well as techniques for selecting a preferred radiation-balanced solution in the first place), but progress is being made towards implementing the method for the gravitational problem. In the meantime, it is instructive to consider how such an equilibrium radiating spacetime could be used.

First, since the spacetime will contain at large distances a superposition of ingoing and outgoing gravitational radiation, we should be able to separate out the outgoing-wave contribution and use it as an approximation for the purely outgoing radiation from the physical system.

Second, a timelike hypersurface of the spacetime could be used as an alternative to post-Newtonian or conformally flat initial data for a full numerical evolution, in one of two ways. First, while the RB spacetime is an equilibrium solution of Einstein's equations, valid for all time, it could still be used to provide initial data for a full numerical evolution, since the outer boundary condition of such an evolution would be one of outgoing rather than balanced radiation, and thus once the 'missing' ingoing radiation had failed to propagate from the outer boundary to the location of the sources, the orbits would begin to decay. However, since that would mean consuming precious supercomputer evolution waiting for the incoming waves to stop, it would be preferable to extend the method of extracting a metric configuration containing only outgoing radiation to cover not just the wave zone, but the entire time slice.

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